SWAT/116
hep-th/9606116
June 1996

The $N = 1$ supersymmetric bootstrap and Lie algebras

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ABSTRACT

The bootstrap programme for finding exact S-matrices of integrable quantum field theories with N=1 supersymmetry is investigated. New solutions are found which have the same fusing data as bosonic theories related to the classical affine Lie algebras. When the states correspond to a spinor spot of the Dynkin diagram they are kinks which carry a non-zero topological charge. Using these results, the S-matrices of the supersymmetric O($2n$) sigma model and sine-Gordon model can be described in a uniform way.

1. Introduction

If a field theory in two-dimensions is integrable then its S-matrix factorizes and the bootstrap programme becomes manageable. Many field theories have been solved exactly by applying the axioms of S-matrix theory in tandem with the bootstrap. It is fascinating that the known solutions of the bootstrap equations are related to (affine) Lie algebras. The bootstrap is characterized by a set of data known as the fusing angles which seem to have a rather universal character since they re-appear in many different theories.

The question that we set ourselves in this paper is whether there is a similar picture for theories with supersymmetry. In particular, we consider the case of $N = 1$ supersymmetry. In the presence of supersymmetry, the bootstrap equations become more complicated since in internal lines one must sum over all states in a super-multiplet. Nevertheless, the solutions which we find have a very characteristic form. Let us suppose that the $i^{\rm th}$ multiplet is labelled $|\xi_i, A_i(\theta)\rangle$, where A_i specifies the supersymmetric quantum numbers, i.e. usually boson or fermion, but sometimes more complicated 'kink' representations appear, and ξ_i specifies any additional quantum numbers needed to label the states.[1] Our S-matrix elements have the form of an ansatz made by Schoutens [1] in which there is no mixing between the supersymmetric and internal quantum numbers. This mean that the S-matrices have the form of a product

$$tildeS_{\xi_i\xi_j\to\xi_j'\xi_i'}(\theta)S_{A_iA_j\to A_j'A_i'}(\theta) = {}_{\rm out}\langle\xi_j', A_j'(\theta_2)\xi_i', A_i'(\theta_1)|\xi_i, A_i(\theta_1)\xi_j, A_j(\theta_2)\rangle_{\rm in}, \quad (1.1)$$

where $\theta = \theta_1 - \theta_2$. In this ansatz, each of the factors separately satisfies the axioms of S-matrix theory: unitarity and crossing. Moreover, $tildeS(\theta)$ is an S-matrix of of purely bosonic theory in its own right and hence satisfies the bootstrap equations with some characteristic set of fusing angles u_{ij}^k.[2] This implies that the supersymmetric part $S(\theta)$ has to satisfy the same bootstrap equations, i.e. with the same fusing angles; however, since the bound-state pole is already present in the factor $tildeS(\theta)$ the supersymmetric factor satisfies the bootstrap equations passively, in the sense that it does not have a pole at the fusing rapidity $\theta = iu_{ij}^k$. In fact, the supersymmetric S-matrix factors that we will construct introduce no additional poles onto the physical strip and hence do not introduce any additional bound-states. This means that the spectrum of states and their fusings follow exactly those of the bosonic theory.

The formalism of supersymmetric factorizable S-matrices was considered some time ago; in particular we refer to the work of Bernard and LeClair [2], Ahn [3] and Schoutens [1]. What was missing from this early work were complete solutions to the bootstrap programme. We shall find that in certain cases, to find consistent solutions of the bootstrap

[1] θ is the rapidity of the state.

[2] In this notation the fusion occurs at the imarginary rapidity difference iu_{ij}^k.

equations we shall have to consider supersymmetric states which carry topological charge. It transpires that these representations are not carried by particle states, rather they are carried by kinks.

2. Minimal supersymmetric S-matrix blocks

In this section we review the construction of the basic S-matrix building blocks describing the interactions of supersymmetric particles and kinks. The question of how to put these blocks together to form a consistent S-matrix for which the bootstrap closes, will be treated in later sections.

Supersymmetry is realized on a super-multiplet of states A_i, where i labels the multiplet

$$\mathcal{Q}|A_i(\theta)\rangle = \sqrt{m_i} e^{\theta/2} Q|A_i(\theta)\rangle, \qquad \bar{\mathcal{Q}}|A_i(\theta)\rangle = \sqrt{m_i} e^{-\theta/2} \bar{Q}|A_i(\theta)\rangle, \tag{2.1}$$

where θ is the rapidity of the state and Q and $\bar{Q}$ are matrices which act on the states of the super-multiplet and which satisfy

$$Q^2 = 1, \qquad \bar{Q}^2 = 1. \tag{2.2}$$

The action of supersymmetry on multiple states involves braiding factors:

$$\begin{aligned}&\mathcal{Q}|A_1(\theta_1)A_2(\theta_2)\cdots A_N(\theta_N)\rangle \\ &= \sum_{k=1}^{N} \sqrt{m_k}\ e^{\theta_k/2} |(Q_L A_1(\theta_1))\cdots(Q_L A_{k-1}(\theta_{k-1}))(Q A_k(\theta_k)) A_{k+1}(\theta_{k+1})\cdots A_N(\theta_N)\rangle.\end{aligned} \tag{2.3}$$

Similarly for $\bar{\mathcal{Q}}$

$$\begin{aligned}&\bar{\mathcal{Q}}|A_1(\theta_1)A_2(\theta_2)\cdots A_N(\theta_N)\rangle \\ &= \sum_{k=1}^{N} \sqrt{m_k}\ e^{-\theta_k/2} |(Q_L A_1(\theta_1))\cdots(Q_L A_{k-1}(\theta_{k-1}))(\bar{Q} A_k(\theta_k)) A_{k+1}(\theta_{k+1})\cdots A_N(\theta_N)\rangle.\end{aligned} \tag{2.4}$$

In the above, Q_L is the fermion parity operator. In particular, for two states we have

$$\begin{aligned}\mathcal{Q}|A_1(\theta_1)A_2(\theta_2)\rangle &= \sqrt{m_1}\ e^{\theta_1/2} Q_1|A_1(\theta_1)A_2(\theta_2)\rangle + \sqrt{m_2}\ e^{\theta_2/2} Q_2|A_1(\theta_1)A_2(\theta_2)\rangle, \\ \bar{\mathcal{Q}}|A_1(\theta_1)A_2(\theta_2)\rangle &= \sqrt{m_1}\ e^{-\theta_1/2} \bar{Q}_1|A_1(\theta_1)A_2(\theta_2)\rangle + \sqrt{m_2}\ e^{-\theta_2/2} \bar{Q}_2|A_1(\theta_1)A_2(\theta_2)\rangle,\end{aligned} \tag{2.5}$$

where $Q_1 = Q\otimes I$, $Q_2 = Q_L \otimes Q$, $\bar{Q}_1 = Q\otimes I$ and $\bar{Q}_2 = Q_L\otimes \bar{Q}$.

A factorizable S-matrix is specified completely by the two-body S-matrix elements which are defined as follows:

$$|A_i(\theta_1)A_j(\theta_2)\rangle_{\rm in} = \sum_{A_i' A_j'} S_{A_iA_j\to A_j' A_i'}(\theta_1-\theta_2)|A_j'(\theta_2)A_i'(\theta_1)\rangle_{\rm out}. \tag{2.6}$$

2.1 The scattering of particles

The supermultiplets are doublets containing a boson $|\phi\rangle$ and a fermion $|\psi\rangle$. A suitable representation for the supercharges, in the basis $\{\phi, \psi\}$, is [1]

$$Q = \begin{pmatrix} 0 & \epsilon \\ \epsilon^* & 0 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & \epsilon^* \\ \epsilon & 0 \end{pmatrix}, \tag{2.7}$$

where $\epsilon = \exp(i\pi/4)$. The fermionic parity operator Q_L is

$$Q_L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.8}$$

It can be easily checked that this representation of supersymmetry carries zero topological charge: $T = \frac{1}{2}\{\bar{Q}, Q\} = 0$.

Here we describe the S-matrix building blocks constructed by Schoutens [1] which describe the scattering of two such particles supermultiplets of masses m_1 and m_2. For a supersymmetric theory the action of supersymmetry commutes with the S-matrix:[3]

$$\mathcal{Q} S_{\mathrm{P}}(\theta) = S_{\mathrm{P}}(\theta)\mathcal{Q}, \qquad \bar{\mathcal{Q}} S_{\mathrm{P}}(\theta) = S_{\mathrm{P}}(\theta)\bar{\mathcal{Q}}. \tag{2.9}$$

We also require that the Yang-Baxter equation is satisfied which ensures that the S-matrix is factorizable.[4] The general solution of Schoutens is then fixed up to an overall scalar function $G(\theta)$ and a constant α.

$$\begin{aligned} S_{\mathrm{P}}(\theta) &= G(\theta)\frac{1}{2i}(Q_1 - Q_2)(\bar{Q}_1 - \bar{Q}_2) \\ &+ \alpha F(\theta)\left(1 - \tanh\left(\frac{\theta + \log(m_1/m_2)}{4}\right) Q_1 Q_2\right)\left(1 + \tanh\left(\frac{\theta - \log(m_1/m_2)}{4}\right) \bar{Q}_1 \bar{Q}_2\right), \end{aligned} \tag{2.10}$$

where

$$F(\theta) = \frac{m_1 + m_2 + 2\sqrt{m_1 m_2}\cosh(\theta/2)}{2i\sinh\theta} G(\theta). \tag{2.11}$$

The constant α measures the strength of Bose-Fermi mixing interactions.

The function $G(\theta)$ is determined by imposing unitarity and crossing symmetry. A minimal solution for the case when the particles are self-conjugate was found in [1]. We shall write down explicit expressions for these factors in section 3.1.

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[3] The subscript P indicates that these S-matrices describe the scattering of particle states, as opposed to kinks described in the next subsection.

[4] The Yang-Baxter equation is non-trivial in this context because there are non-diagonal processes; for instance the boson can reflect off the fermion.

2.2 *The scattering of kinks*

If there are topologically charged states in the theory, then the supersymmetry algebra is modified by central charges [4]. In general, states which carry topological charges are kinks $K_{ab}(\theta)$ which interpolate between two vacua $a, b \in \Gamma$, where Γ is the set of vacua. It is important to realize that kink states are *not* in general equivalent to a set of particle states, due to fact that multi-kink states must respect an adjacency condition, i.e. $|\cdots K_{ab}(\theta_1)K_{cd}(\theta_2)\cdots\rangle$ requires $b = c$.

An S-matrix with supersymmetry can be associated to a system whose fundamental excitations are four kinks, interpolating between three vacua, labelled by 0, $\frac{1}{2}$, 1. In the basis $\{K_{0\frac{1}{2}}, K_{1\frac{1}{2}}, K_{\frac{1}{2}0}, K_{\frac{1}{2}1}\}$, and appropriate representation of supersymmetry is

$$Q = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{2.12}$$

The topological charges are

$$T = \frac{1}{2}\{Q, \bar{Q}\} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{2.13}$$

so $K_{0\frac{1}{2}}$ and $K_{1\frac{1}{2}}$ have $T = 1$, and $K_{\frac{1}{2}0}$ and $K_{\frac{1}{2}1}$ have $T = -1$.

The S-matrix describing the scattering of the kinks can be written as [1]

$$S_{\mathrm{K}}(\theta) = K(\theta)(\cosh(\gamma\theta) - \sinh(\gamma\theta)Q_1\bar{Q}_1)(\cosh(\theta/4) - \sinh(\theta/4)Q_1Q_2), \tag{2.14}$$

where $\gamma = \log 2/2\pi i$. We will denote the explicit S-matrix elements for $K_{ab}(\theta_1)+K_{bc}(\theta_2) \to K_{ad}(\theta_2) + K_{dc}(\theta_1)$ as

$$S\left(\begin{array}{cc} a & d \\ b & c \end{array}\middle| \theta_1 - \theta_2\right). \tag{2.15}$$

Explicitly, the non-zero elements are

$$\begin{aligned}
S\left(\begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array}\middle| \theta\right) &= S\left(\begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array}\middle| \theta\right) = K(\theta)2^{(i\pi-\theta)/2\pi i}\cos\left(\frac{\theta}{4i} - \frac{\pi}{4}\right) \\
S\left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array}\middle| \theta\right) &= S\left(\begin{array}{cc} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{array}\middle| \theta\right) = K(\theta)2^{\theta/2\pi i}\cos\left(\frac{\theta}{4i}\right) \\
S\left(\begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array}\middle| \theta\right) &= S\left(\begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array}\middle| \theta\right) = K(\theta)2^{(i\pi-\theta)/2\pi i}\cos\left(\frac{\theta}{4i} + \frac{\pi}{4}\right) \\
S\left(\begin{array}{cc} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{array}\middle| \theta\right) &= S\left(\begin{array}{cc} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{array}\middle| \theta\right) = K(\theta)2^{\theta/2\pi i}\cos\left(\frac{\theta}{4i} - \frac{\pi}{2}\right)
\end{aligned} \tag{2.16}$$

The scalar scalar function $K(\theta)$ is determined by crossing symmetry and unitarity. The minimal solution for $K(\theta)$ is

$$K(\theta) = \frac{1}{\sqrt{\pi}} \prod_{k=1}^{\infty} \frac{\Gamma(k - \frac{1}{2} + \theta/2\pi i)\Gamma(k - \theta/2\pi i)}{\Gamma(k + \theta/2\pi i)\Gamma(k + \frac{1}{2} - \theta/2\pi i)}. \tag{2.17}$$

This S-matrix was first written down by Zamolodchikov [5] to describe the perturbation of the $c = 7/10$ superconformal minimal model by an operator of dimension $6/5$. Subsequently it was realized that the S-matrix elements are related to a solution of the Yang-Baxter equation associated to the quantum group $U_q(su(2))$ [6].

3. The fusing procedure

In the last section we described the construction of supersymmetric S-matrix blocks describing particles and kinks. In this section we show how such blocks can be put together to describe supersymmetric theories with rich spectra of states.

As it stands the supersymmetric S-matrices that we have constructed in the last section are 'still-born' in the sense that they have no bound-state poles on the physical strip. In order to rectify this we can take, following Schoutens [1], an ansatz which consists of a product of the S-matrix of purely bosonic theory $S'(\theta)$ with the supersymmetric S-matrix blocks of the last section which we denote as $S(\theta)$. The resulting S-matrix is then schematically of the form

$$S_{\text{SUSY}}(\theta) = S'(\theta) \otimes S(\theta). \tag{3.1}$$

This notation is short-hand for the product form displayed in (1.1). These S-matrices describe a set of states carrying the quantum numbers of the bosonic theory as well as forming either particle or kink supermultiplets. An ansatz of this form manifestly satisfies unitarity and crossing symmetry, because each of the factors does so separately. The only non-trivial consistency conditions arise from the existence of bound-states due to poles in the bosonic factor. By itself the bosonic part, of course, satisfies the required bootstrap equations; however, what has to be checked is that the supersymmetric part does not introduce any additional poles onto the physical strip and satisfies the *same* bootstrap equations, although only passively, in the sense that on its own it does not have the required bound-state poles. This implies that in our discussion in this section the expression for the supersymmetric part of the bound-state wavefunction is given by the limit of the two particle state, at the position of the pole in the bosonic part, rather than the residue. Let us suppose that the fusing angles of the bosonic theory are u_{ij}^k. The bootstrap equations for $S(\theta)$ can be summed up by the relation

$$|A_i(\theta + i\bar{u}_{i\bar{k}}^{\bar{j}})A_j(\theta - i\bar{u}_{j\bar{k}}^{\bar{i}})\rangle = \sum_{A_k} f_{A_i A_j}^{A_k} |A_k(\theta)\rangle, \tag{3.2}$$

where the sum is over all states in the $k^{\rm th}$ super-multiplet. In the above we have defined $\bar{u} = \pi - u$ and $\bar{j}$ is the charge conjugate multiplet of j. The coupling constants are related to the S-matrix evaluated at the fusing rapidity:

$$S_{A_iA_j\to A'_jA'_i}(iu_{ij}^k) = \sum_{A_k} f_{A_iA_j}^{A_k}\left(f_{A'_iA'_j}^{A_k}\right)^*. \tag{3.3}$$

Notice that in the above formula we do not relate the coupling constants to the residue of the S-matrix as one does for the bosonic part of the S-matrix, since $S(\theta)$ has no pole.

Our task in this section is to show in a general context how the supersymmetric blocks solve the bootstrap equations with the same fusing angles encountered in the bosonic theories. In the section 4 and 5 we go on to construct explicitly supersymmetric S-matrices by consider particular bosonic factors $tildeS(\theta)$ and their fusing structure.

3.1 Fusing between particles

In this section we consider the bound-states that can appear in the S-matrix $S_{\rm P}(\theta)$ describing the scattering of supersymmetric particles. We denote the mass of the a^{rmth} super-multiplet m_a. If the bosonic part of the S-matrix provides a bound-state pole at $\theta = iu_{ab}^c$ in the scattering of a with b then the bootstrap equations for the supersymmetric part of the S-matrix are equivalent to defining the wavefunction of the bound-state as:

$$\begin{aligned}|\phi_c(\theta)\rangle &= \frac{1}{(f_{\phi\phi})_{ab}^c}|\phi_a(\theta+i\bar{u}_{a\bar{c}}^{\bar{b}})\phi_b(\theta-i\bar{u}_{b\bar{c}}^{\bar{a}})\rangle = \frac{1}{(f_{\psi\psi})_{ab}^c}|\psi_a(\theta+i\bar{u}_{a\bar{c}}^{\bar{b}})\psi_b(\theta-i\bar{u}_{b\bar{c}}^{\bar{a}})\rangle \\ |\psi_c(\theta)\rangle &= \frac{1}{(f_{\phi\psi})_{ab}^c}|\phi_a(\theta+i\bar{u}_{a\bar{c}}^{\bar{b}})\psi_b(\theta-i\bar{u}_{b\bar{c}}^{\bar{a}})\rangle = \frac{1}{(f_{\psi\phi})_{ab}^c}|\psi_a(\theta+i\bar{u}_{a\bar{c}}^{\bar{b}})\phi_b(\theta-i\bar{u}_{b\bar{c}}^{\bar{a}})\rangle,\end{aligned} \tag{3.4}$$

where $\bar{a}$ denotes the charge conjugate of a and the coupling constants are related to the S-matrix elements via (3.3) from which one deduces that

$$\begin{aligned}(f_{\phi\phi})_{ab}^c &= \sqrt{S_{\phi\phi\to\phi\phi}^{[ab]}(iu_{ab}^c)}, &\quad (f_{\psi\psi})_{ab}^c &= \sqrt{S_{\psi\psi\to\psi\psi}^{[ab]}(iu_{ab}^c)} \\ (f_{\phi\psi})_{ab}^c &= \sqrt{S_{\phi\phi\to\psi\psi}^{[ab]}(iu_{ab}^c)}, &\quad (f_{\psi\phi})_{ab}^c &= \sqrt{S_{\psi\psi\to\phi\phi}^{[ab]}(iu_{ab}^c)}.\end{aligned} \tag{3.5}$$

From this ift follows

$$\frac{(f_{\psi\psi})_{ab}^c}{(f_{\phi\phi})_{ab}^c} = \sqrt{\frac{m_a+m_b-m_c}{m_a+m_b+m_c}},\ \frac{(f_{\phi\psi})_{ab}^c}{(f_{\phi\phi})_{ab}^c} = \sqrt{\frac{m_b-m_a+m_c}{m_a+m_b+m_c}},\ \frac{(f_{\psi\phi})_{ab}^c}{(f_{\phi\phi})_{ab}^c} = \sqrt{\frac{m_a-m_b+m_c}{m_a+m_b+m_c}}. \tag{3.6}$$

The definitions of these bound-states are consistent with the action of supersymmetry [1].

These equations are highly restrictive and determine the coupling constant α [1]:

$$\alpha = -\frac{\sin(u_{ab}^c)}{m_c} = -\frac{\sqrt{4m_a^2m_b^2-(m_a^2+m_b^2-m_c^2)^2}}{2m_am_bm_c}, \tag{3.7}$$

which in turn places a highly restrictive constraint on the allowed fusing rules and mass spectrum in the theory. The spectrum is fixed up to a constant H:

$$m_a = m\sin(a\pi/H), \quad a = 1, 2, \ldots, n, \tag{3.8}$$

where the total number of particles n will depend on H. The allowed fusings depend upon whether the particles are self-conjugate or not. In the former case we have $n = [H/2]$ and the only fusing angles which are consistent with the mass spectrum (for generic values of H) are

$$u_{ab}^{a+b} = \frac{(a+b)\pi}{H} \quad (a+b \le n), \qquad u_{ab}^{|b-a|} = \pi - \frac{|b-a|\pi}{H}. \tag{3.9}$$

For some theories considered in the next section, H will be constrained to some integer value and there exist additional fusing angles. Also in one class of solutions to the bootstrap, we shall find the particles are not self-conjugate. These points will be discussed fully in section 4. For the present, we will work out the implicitions of the fusing rules in (3.9).

It is straightforward to show that the crucial condition (3.7) is satisfied for each of the fusings (3.9) and the value of the coupling constant is fixed to be $\alpha^{-1} = -m$. The S-matrix elements describing the scattering of particle supermultiplets are explicitly

$$\begin{aligned}
S^{[ab]}_{\phi\phi\to\phi\phi}(\theta) &= \left(1 + \frac{2\sin(\frac{a+b}{2H}\pi)\cos(\frac{a-b}{2H}\pi)}{\sin(\frac{\theta}{i})}\right) G^{[ab]}(\theta),\\
S^{[ab]}_{\psi\psi\to\psi\psi}(\theta) &= \left(-1 + \frac{2\sin(\frac{a+b}{2H}\pi)\cos(\frac{a-b}{2H}\pi)}{\sin(\frac{\theta}{i})}\right) G^{[ab]}(\theta),\\
S^{[ab]}_{\phi\phi\to\psi\psi}(\theta) = S^{[ab]}_{\psi\psi\to\phi\phi}(\theta) &= \frac{\sqrt{\sin(\frac{a\pi}{H})\sin(\frac{b\pi}{H})}}{\cos(\frac{\theta}{2i})} G^{[ab]}(\theta),\\
S^{[ab]}_{\phi\psi\to\phi\psi}(\theta) = S^{[ab]}_{\psi\phi\to\psi\phi}(\theta) &= \frac{\sqrt{\sin(\frac{a\pi}{H})\sin(\frac{b\pi}{H})}}{\sin(\frac{\theta}{2i})} G^{[ab]}(\theta),\\
S^{[ab]}_{\phi\psi\to\psi\phi}(\theta) &= \left(1 - \frac{2\sin(\frac{a-b}{2H}\pi)\cos(\frac{a+b}{2H}\pi)}{\sin(\frac{\theta}{i})}\right) G^{[ab]}(\theta),\\
S^{[ab]}_{\psi\phi\to\phi\psi}(\theta) &= \left(1 + \frac{2\sin(\frac{a-b}{2H}\pi)\cos(\frac{a+b}{2H}\pi)}{\sin(\frac{\theta}{i})}\right) G^{[ab]}(\theta).
\end{aligned} \tag{3.10}$$

We must now verify that the S-matrix elements above satisfy the bootstrap equations which follow from the fusing rules (3.9). First of all let us derive the expressions for the unitarizing/crossing factors $G^{[ab]}(\theta)$. We will assume for the remainder of this section that the particles are self-conjugate. The unitarity and crossing symmetry constraints for the S-matrix elements (3.10) require

$$\begin{aligned}
G^{[ab]}(\theta)G^{[ab]}(-\theta) &= \frac{\sin^2(\frac{\theta}{2i})\cos^2(\frac{\theta}{2i})}{\sin(\frac{\theta}{2i}+\frac{(a+b)\pi}{2H})\sin(\frac{\theta}{2i}-\frac{(a+b)\pi}{2H})\cos(\frac{\theta}{2i}+\frac{(a-b)\pi}{2H})\cos(\frac{\theta}{2i}-\frac{(a-b)\pi}{2H})},\\
G^{[ab]}(\theta) &= G^{[ab]}(i\pi-\theta).
\end{aligned} \tag{3.11}$$

It is useful to rewrite the second equation above using the first as

$$G^{[ab]}(i\pi+\theta)G^{[ab]}(i\pi-\theta) \\ = \frac{\sin^2(\frac{\theta}{2i})\cos^2(\frac{\theta}{2i})}{\sin(\frac{\theta}{2i}+\frac{(a+b)\pi}{2H})\sin(\frac{\theta}{2i}-\frac{(a+b)\pi}{2H})\cos(\frac{\theta}{2i}+\frac{(a-b)\pi}{2H})\cos(\frac{\theta}{2i}-\frac{(a-b)\pi}{2H})}. \tag{3.12}$$

Solving the above system for $G^{[ab]}(\theta)$ we find:

$$\begin{aligned} G^{[ab]}(\theta) &= R^{[ab]}(\theta)R^{[ab]}(i\pi-\theta), \\ R^{[ab]}(\theta) &= \frac{1}{\Gamma(\frac{\theta}{2\pi i})\Gamma(\frac{\theta}{2\pi i}+\frac{1}{2})}\prod_{k=1}^{\infty}\frac{\Gamma(\frac{\theta}{2\pi i}+\frac{a+b}{2H}+k-1)\Gamma(\frac{\theta}{2\pi i}-\frac{a+b}{2H}+k)}{\Gamma(\frac{\theta}{2\pi i}+\frac{a+b}{2H}+k-\frac{1}{2})\Gamma(\frac{\theta}{2\pi i}-\frac{a+b}{2H}+k+\frac{1}{2})} \\ &\quad\times\frac{\Gamma(\frac{\theta}{2\pi i}+\frac{a-b}{2H}+k-\frac{1}{2})\Gamma(\frac{\theta}{2\pi i}-\frac{a-b}{2H}+k-\frac{1}{2})}{\Gamma(\frac{\theta}{2\pi i}+\frac{a-b}{2H}+k)\Gamma(\frac{\theta}{2\pi i}-\frac{a-b}{2H}+k)}. \end{aligned} \tag{3.13}$$

This solution follows from the general expression in [1]. These expressions are not unique since they are subject to the usual CDD ambiguities. However, they are the minimal solutions: the ones with the smallest number of poles and zeros on the physical strip. One can verify directly that $G^{[ab]}(\theta)$ introduces no poles onto the physical strip, so as we have already mentioned the supersymmetric part of the S-matrix does not introduce any additional bound-state poles.

We must now verify that these S-matrix elements satisfy the bootstrap equations for the fusing angles in (3.9). Schematically these equations are of the form

$$S_{\rm P}^{[dc]}(\theta)\sim S_{\rm P}^{[da]}(\theta-i\bar{u}_{a\bar{c}}^{\bar{b}})S_{\rm P}^{[db]}(\theta+i\bar{u}_{b\bar{c}}^{\bar{a}}). \tag{3.14}$$

Here, $\bar{a}$ denotes the charge conjugate of a, although in the present section we are assuming $\bar{a}=a$. More explicitly a given S-matrix element can be obtained in two different ways, for example for the fusing $c=a+b$:

$$\begin{aligned} &S^{[dc]}_{\phi\phi\to\phi\phi}(\theta) \\ &= S^{[da]}_{\phi\phi\to\phi\phi}(\theta-i\bar{u}_{a\bar{c}}^{\bar{b}})S^{[db]}_{\phi\phi\to\phi\phi}(\theta+i\bar{u}_{b\bar{c}}^{\bar{a}})+\frac{(f_{\psi\psi})_{ab}^{c}}{(f_{\phi\phi})_{ab}^{c}}S^{[da]}_{\phi\phi\to\phi\phi}(\theta-i\bar{u}_{a\bar{c}}^{\bar{b}})S^{[db]}_{\psi\phi\to\psi\phi}(\theta+i\bar{u}_{b\bar{c}}^{\bar{a}}) \\ &= S^{[da]}_{\phi\psi\to\psi\phi}(\theta-i\bar{u}_{a\bar{c}}^{\bar{b}})S^{[db]}_{\phi\psi\to\psi\phi}(\theta+i\bar{u}_{b\bar{c}}^{\bar{a}})+\frac{(f_{\phi\phi})_{ab}^{c}}{(f_{\psi\psi})_{ab}^{c}}S^{[da]}_{\phi\psi\to\phi\psi}(\theta-i\bar{u}_{a\bar{c}}^{\bar{b}})S^{[db]}_{\psi\psi\to\phi\phi}(\theta+i\bar{u}_{b\bar{c}}^{\bar{a}}). \end{aligned} \tag{3.15}$$

By writting down all 16 of the possible fusings one can directly verify that the bootstrap equations are consistent with supersymmetry and one finds the following fusing relation for the unitarizing/crossing factors:

$$G^{[d,a+b]}(\theta)=\xi_{ab}^{d}(\theta)G^{[da]}\left(\theta-\frac{i\pi b}{H}\right)G^{[db]}\left(\theta+\frac{i\pi a}{H}\right). \tag{3.16}$$

with

$$\xi^{d}_{ab}(\theta) = \frac{\sin(\frac{\theta}{i})\left[\sin(\frac{\theta}{i} + \frac{(a-b)\pi}{H}) + \sin(\frac{d\pi}{H})\right]}{\sin(\frac{\theta}{i} + \frac{a\pi}{H})\sin(\frac{\theta}{i} - \frac{b\pi}{H})}. \tag{3.17}$$

It is a tedious but straightforward excerise to verify that the solution in (3.13) satisfies the fusion equation (3.16).

The bootstrap equations arising from the second fusion angle in (3.9) can be checked in a similar way. Following the same strategy as before, we find that each one of the new sixteen bootstrap equations is satisfied if (we take $b \geq a$)

$$G^{[d,b-a]}(\theta) = \xi^{-d}_{-a,b}(\theta) G^{[db]}\left(\theta - \frac{i\pi a}{H}\right) G^{[da]}\left(\theta + \frac{i\pi(H-b)}{H}\right). \tag{3.18}$$

This follows from (3.16) using crossing symmetry and unitarity.

To summarize the results of this section, we have demonstrated that the S-matrix describing the scattering of particle super-multiplets satisfies bootstrap equations which are schematically of the form

$$\begin{aligned} &\text{(i)} \qquad S_{\mathrm{P}}^{[d,a+b]}(\theta) \sim S_{\mathrm{P}}^{[da]}\left(\theta - \frac{i\pi b}{H}\right) S_{\mathrm{P}}^{[db]}\left(\theta + \frac{i\pi a}{H}\right) \\ &\text{(ii)} \qquad S_{\mathrm{P}}^{[d,b-a]}(\theta) \sim S_{\mathrm{P}}^{[da]}\left(\theta - \frac{i\pi a}{H}\right) S_{\mathrm{P}}^{[db]}\left(\theta + \frac{i\pi(H-b)}{H}\right), \end{aligned} \tag{3.19}$$

where in the second equation $b \geq a$.

3.2 Fusing between kinks

It is natural at this stage to ask what happens if the bosonic theory contains states whose mass does not respect the crucial condition (3.7) at the fusing points? These states cannot correspond to ordinary particle super-multiplets if the fusing rules of the bosonic theory are to be respected. We shall find that in these situations the states must be kink super-multiplets of the type discussed in section 2.2.

The construction of a minimal supersymmetric S-matrix for the scattering of kinks, $\{K_{0\frac{1}{2}}, K_{1\frac{1}{2}}, K_{\frac{1}{2}0}, K_{\frac{1}{2}1}\}$ was described in section 2.2. The S-matrix described there does not have any poles on the physical strip and so as it stands there are no kink-kink bound-states. However, the bosonic part of the S-matrix will introduce poles and so we must consider the possibility of kink-kink bound-states. Notice that all the possible two-kink states carry zero topological charge and hence it natural to suppose that the bound-states of kinks correspond to particles. In the following, we shall indicate how such an indentification is consistent with the action of supersymmetry and furthermore the scattering of the bound-states, as deduced from the bootstrap equations, is identical to that for the particle super-multiplets written down in section 2.1.

Suppose that the fusing rules imply that two kinks form a bound-state at a rapidity difference $i\xi$, with $0<\xi<\pi$. The bound-state is identified with a particle super-multiplet (ϕ,ψ) of mass

$$2m\cos(\xi/2). \tag{3.20}$$

It is important that the fusion is consistent for *any* value of ξ, subject to the constraint that the pole is on the physical strip. Thus it is possible that a series of particle multiplets can appear as bound-state of two kinks. Our full solutions of the bootstrap equations will have this property.

The coupling constants for these processes are defined via:

$$|K_{ab}(\theta+i\xi/2)K_{bc}(\theta-i\xi/2)\rangle = f^{\phi}_{abc}|\phi(\theta)\rangle + f^{\psi}_{abc}|\psi(\theta)\rangle. \tag{3.21}$$

The non-zero coupling constants are

$$\begin{aligned} f^{\phi}_{0\frac12 0} = f^{\phi}_{1\frac12 1} = 2^{(\pi-2\xi)/4\pi} f^{\phi}_{\frac12 0\frac12} = 2^{(\pi-2\xi)/4\pi} f^{\phi}_{\frac12 1\frac12} &= \sqrt{K(i\xi)2^{(\pi-\xi)/2\pi}\cos\left(\frac{\xi-\pi}{4}\right)} \\ f^{\psi}_{1\frac12 0} = -f^{\psi}_{0\frac12 1} = 2^{(\pi-2\xi)/4\pi} i f^{\psi}_{\frac12 0\frac12} = -2^{(\pi-2\xi)/4\pi} i f^{\psi}_{\frac12 1\frac12} &= \sqrt{K(i\xi)2^{(\pi-\xi)/2\pi}\cos\left(\frac{\xi+\pi}{4}\right)}. \end{aligned} \tag{3.22}$$

One can easily verify that the relation between the coupling constants (3.22) and the S-matrix elements in (3.3) is satisfied.

The coupling constants in (3.22) are consistent with the action of supersymmetry and fermionic parity. For example

$$\begin{aligned} \mathcal{Q}|\phi(\theta)\rangle &= \frac{\sqrt{m}}{f^{\phi}_{0\frac12 0}}(Q_1+Q_2)|K_{0\frac12}(\theta+i\xi/2)K_{\frac12 0}(\theta-i\xi/2)\rangle \\ &= \frac{\sqrt{m}e^{\theta/2}}{f^{\phi}_{0\frac12 0}}\left(-ie^{i\xi/4}+e^{-i\xi/4}\right)|K_{1\frac12}(\theta+i\xi/2)K_{\frac12 0}(\theta-i\xi/2)\rangle \\ &= \frac{e^{-i\pi/4}e^{\theta/2}\sqrt{2m\cos(\xi/2)}}{f^{\psi}_{1\frac12 0}}|K_{1\frac12}(\theta+i\xi/2)K_{\frac12 0}(\theta-i\xi/2)\rangle \\ &= e^{-i\pi/4}e^{\theta/2}\sqrt{2m\cos(\xi/2)}|\psi(\theta)\rangle, \end{aligned} \tag{3.23}$$

which is exactly the required transformation for a particle of mass (3.20).

As we alluded to earlier, the bound-state structure is consistent for any value of ξ. In the theories that we shall construct in following sections each of the particle multiplets with masses parameterized as in (3.8) will appear as a kink-kink bound-state for different values of ξ. By comparing (3.20) with (3.8) we see that for the a^{th} particle multiplet we have $\xi=\pi-2\pi a/H$. The scattering of these states can then be determined by the bootstrap equations. Notice that since a given state, say ϕ_a, can be obtained in a number of different ways, for example as a bound-state of $K_{0\frac12}$ with $K_{\frac12 0}$ or $K_{1\frac12}$ with $K_{\frac12 1}$. Hence

the scattering amplitudes of ϕ_a which follow from the bootstrap equations can be obtained in different ways. This leads to a highly non-trivial check of the whole construction. Not only do the coupling (3.22) pass this check but the resulting amplitudes for the particles are exactly those that we wrote down in section 3.1

To illustrate the construction of particle scattering amplitudes from the kink amplitudes via the bootstrap equations consider the process $\phi_a\phi_b \to \phi_b\phi_a$ computed by using the fusion

$$|K_{0\frac{1}{2}}(\theta + i\xi/2)K_{\frac{1}{2}0}(\theta - i\xi/2)\rangle = \sqrt{K(i\xi)2^{(\pi-\xi)/2\pi}\cos\left(\frac{\xi-\pi}{4}\right)}|\phi(\theta)\rangle. \tag{3.24}$$

with $\xi = \pi - 2\pi a/H$ for ϕ_a and $\xi = \pi - 2\pi b/H$ for ϕ_b, respectively. The bootstrap equation is illustrated in figure 1.

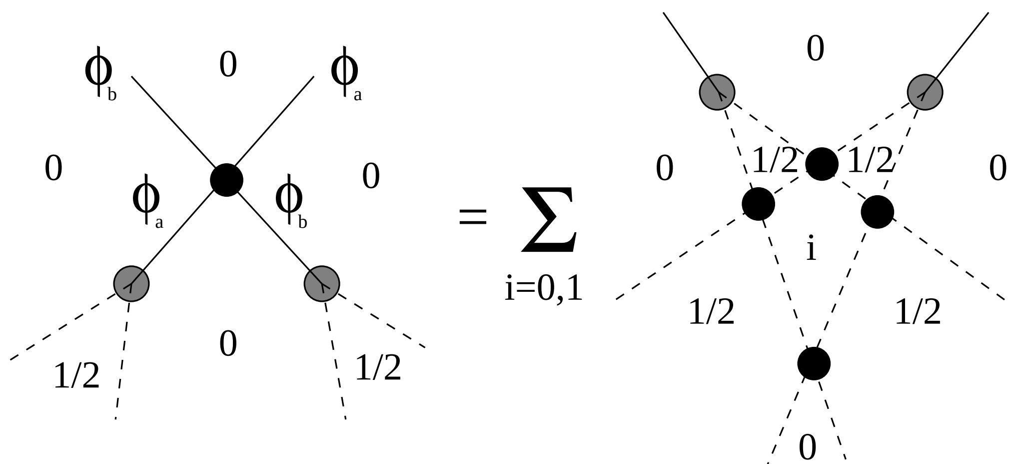

Figure 1. Particle S-matrix element in terms of kink S-matrices

Hence the scattering amplitude for ϕ_a with ϕ_b is equal to

$$\begin{aligned}
&S\left(\begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array}\middle|\,\theta + ix\right) S\left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array}\middle|\,\theta - iy\right) S\left(\begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array}\middle|\,\theta - ix\right) S\left(\begin{array}{cc} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{array}\middle|\,\theta + iy\right) \\
&+ S\left(\begin{array}{cc} 1 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{array}\middle|\,\theta + ix\right) S\left(\begin{array}{cc} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{array}\middle|\,\theta - iy\right) S\left(\begin{array}{cc} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{array}\middle|\,\theta - ix\right) S\left(\begin{array}{cc} \frac{1}{2} & 0 \\ 1 & \frac{1}{2} \end{array}\middle|\,\theta + iy\right) \\
&= K(\theta + ix)K(\theta - iy)K(\theta - ix)K(\theta + iy)\left(\frac{1}{2}\sin(\theta/i) + \cos(y/2)\cos(x/2)\right),
\end{aligned} \tag{3.25}$$

where $x = (a-b)\pi/H$ and $y = \pi - (a+b)\pi/H$. This is precisely equal to $S^{[ab]}_{\phi\phi\to\phi\phi}(\theta)$ in (3.10) by virtue of the following the identity between the unitarizing/crossing factors:

$$G^{[ab]}(\theta) = \frac{\sin(\theta/i)}{2}K(\theta + ix)K(\theta - iy)K(\theta - ix)K(\theta + iy) \tag{3.26}$$

The S-matrix elements for the scattering of kinks with particles also follows by applying the bootstrap equations. For example consider the process $\phi_a(\theta_1) + K_{0\frac{1}{2}}(\theta_2) \to$

$K_{0\frac{1}{2}}(\theta_2) + \phi_a(\theta_1)$. This S-matrix element can be deduced from the bootstrap equation illustrated in figure 2 and is equal to

$$\frac{f^{\phi_a}_{\frac{1}{2}0\frac{1}{2}}}{f^{\phi_a}_{0\frac{1}{2}0}} S\left(\begin{matrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{matrix}\middle|\theta + ix\right) S\left(\begin{matrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{matrix}\middle|\theta - ix\right) + \frac{f^{\phi_a}_{\frac{1}{2}1\frac{1}{2}}}{f^{\phi_a}_{0\frac{1}{2}0}} S\left(\begin{matrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{matrix}\middle|\theta + ix\right) S\left(\begin{matrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{matrix}\middle|\theta - ix\right), \tag{3.27}$$

where $x = \pi/2 - \pi a/H$ and the coupling constants are given in (3.22) with $\xi = \pi - 2\pi a/H$.

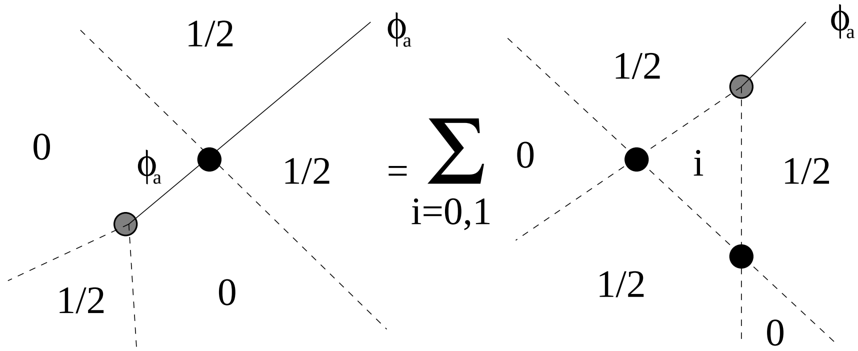

Figure 2. Bootstrap gives particle-kink scattering

3.3 *Fusing between particles and kinks*

Since two kinks can fuse at a rapidity difference $i\xi$ to form a particle, it is also possible for a particle and kink to fuse to form a kink at a rapidity difference $i\pi - i\xi/2$. This allows to introduce the particle-kink coupling constants: The coupling constants for these processes are defined via:

$$|\varphi(\theta + i\xi/2)K_{ab}(\theta - i\pi + i\xi)\rangle = f^{c}_{\varphi ab}|K_{cb}(\theta)\rangle, \tag{3.28}$$

where $\varphi \in (\phi, \psi)$. The non-zero coupling constants are

$$f^{0}_{\phi 0\frac{1}{2}} = f^{1}_{\phi 1\frac{1}{2}} = 2^{(\pi-2\xi)/4\pi} f^{\frac{1}{2}}_{\phi\frac{1}{2}0} = 2^{(\pi-2\xi)/4\pi} f^{\frac{1}{2}}_{\phi\frac{1}{2}1} = \sqrt{K(i\xi)2^{(\pi-\xi)/2\pi}\cos\left(\frac{\xi-\pi}{4}\right)}$$
$$f^{0}_{\psi 1\frac{1}{2}} = -f^{1}_{\psi 0\frac{1}{2}} = 2^{(\pi-2\xi)/4\pi} i f^{\frac{1}{2}}_{\psi\frac{1}{2}1} = -2^{(\pi-2\xi)/4\pi} i f^{\frac{1}{2}}_{\psi\frac{1}{2}0} = \sqrt{K(i\xi)2^{(\pi-\xi)/2\pi}\cos\left(\frac{\xi+\pi}{4}\right)}. \tag{3.29}$$

Notice that the coupling constant are simply the the crossed versions of those in (3.22). It is furthermore possible to verify by explicity constructing all the particle-kink S-matrix elements as illustrated at the end of the last section, that the relation between the coupling constants (3.29) and the S-matrix elements in (3.3) is satisfied. It is also possible to show that the bound-states have the correct transformation properties under supersymmetry and fermionic-parity for *any* value of ξ.

The fusions in (3.29) imply a set of bootstrap equations. It is at this stage that the bootstrap equations close in on themselves because if a kink can be formed as a bound-state of a particle and a kink and a particle can be formed as a bound-state of two kinks, then it follows that a kink can be formed as a bound-state of three kinks. Therefore the resulting bootstrap equations are a highly non-trivial set of consistency conditions on the kink S-matrix elements. To illustrate this, consider the bootstrap equation depicted in figure 3.

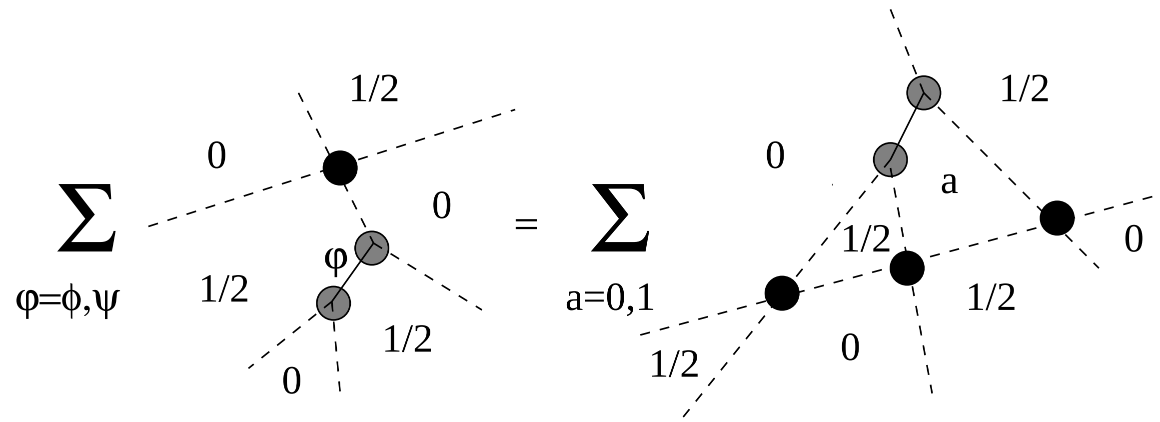

Figure 3. Non-trivial bootstrap equation for kinks

This requires the following relation between the kink S-matrix elements:

$$
\begin{aligned}
S\left(\begin{array}{cc|c} 0 & \frac{1}{2} & \theta \\ \frac{1}{2} & 0 & \end{array}\right) = \alpha(\xi) S\left(\begin{array}{cc|c} 0 & \frac{1}{2} & \theta_1 \\ \frac{1}{2} & 0 & \end{array}\right) S\left(\begin{array}{cc|c} \frac{1}{2} & 0 & \theta_2 \\ 0 & \frac{1}{2} & \end{array}\right) S\left(\begin{array}{cc|c} 0 & \frac{1}{2} & \theta_3 \\ \frac{1}{2} & 0 & \end{array}\right) \\
+ \beta(\xi) S\left(\begin{array}{cc|c} 0 & \frac{1}{2} & \theta_1 \\ \frac{1}{2} & 0 & \end{array}\right) S\left(\begin{array}{cc|c} \frac{1}{2} & 1 & \theta_2 \\ 0 & \frac{1}{2} & \end{array}\right) S\left(\begin{array}{cc|c} 1 & \frac{1}{2} & \theta_3 \\ \frac{1}{2} & 0 & \end{array}\right)
\end{aligned}
\tag{3.30}
$$

where $\theta_1 = \theta - i\xi$, $\theta_2 = \theta$ and $\theta_3 = \theta + i\pi - i\xi$. The quantities $\alpha(\xi)$ and $\beta(\xi)$ are the following functions of the coupling constants:

$$
\begin{aligned}
\alpha(\xi) &= \frac{f^{\phi}_{0\frac{1}{2}0} f^{0}_{\phi(0\frac{1}{2})}}{f^{\phi}_{\frac{1}{2}0\frac{1}{2}} f^{\frac{1}{2}}_{\phi\frac{1}{2}0} + f^{\psi}_{\frac{1}{2}0\frac{1}{2}} f^{\frac{1}{2}}_{\psi\frac{1}{2}0}} = \frac{\sqrt{2}\cos((\xi-\pi)/4)}{\cos(\xi/4)} 2^{-\xi/\pi} \\
\beta(\xi) &= \frac{f^{\psi}_{0\frac{1}{2}0} f^{0}_{\psi 1\frac{1}{2}}}{f^{\phi}_{\frac{1}{2}0\frac{1}{2}} f^{\frac{1}{2}}_{\phi\frac{1}{2}0} + f^{\psi}_{\frac{1}{2}0\frac{1}{2}} f^{\frac{1}{2}}_{\psi\frac{1}{2}0}} = \frac{\sqrt{2}\cos((\xi+\pi)/4)}{\cos(\xi/4)} 2^{-\xi/\pi}.
\end{aligned}
\tag{3.31}
$$

It is a tedious exercise to verify that the S-matrix elements of the kinks do indeed satisfy (3.30) and similar equations that result from considering other possible fusions. The calculation is greatly aided by noting that

$$
K(\theta - i\xi) K(\theta + i\pi - i\xi) = \frac{1}{\cos((\theta - i\xi)/2i)}, \tag{3.32}
$$

which follows from using the unitarity and crossing properties of $K(\theta)$. It is important to realize that these bootstrap equations are satisfied for any value of the fusing angle ξ.

4. Supersymmetric S-matrices associated to Lie algebras

The simplest series of bosonic factorizable S-matrices are purely elastic—the particles carry no internal quantum numbers—and are related to Lie algebra data. In the case of a simply-laced algebra there is a so-called minimal S-matrix which does not depend on any coupling constant. By introducing specific CDD factors depending on a single constant these S-matrices describe the scattering of particle states in affine Toda field theories. There is an elegant group theoretical way to write these S-matrices due to Dorey [7] For the non-simply-laced algebras, the minimal S-matrix itself depends on a coupling constant and they directly describe affine Toda field theories [8]. Below, we consider each of the classical affine Lie algebras in turn and how a supersymmetric S-matrix can be associated to each of them. For these S-matrices the factorization of the S-matrix implied by (1.1) is somewhat redundant since the bosonic factor can be absorbed as a CDD factor into the definition of the supersymmetric factor; however, the factorized form makes the fusion structure of the theories manifest.

Before doing so, it is useful to introduce the following standard notation used for the bosonic theories [9]:

$$\begin{aligned}(x) &= \frac{\sin\left(\frac{\theta}{2i}+\frac{\pi x}{2h}\right)}{\sin\left(\frac{\theta}{2i}-\frac{\pi x}{2h}\right)}, \qquad \{x\} = (x-1)(x+1),\\ \{x\}_v &= \frac{(x-vB-1)(x+vB+1)}{(x+vB+B-1)(x-vB-B+1)}, \qquad [x]_v = \{x\}_v\{h-x\}_v.\end{aligned} \tag{4.1}$$

4.1 The case $a_{n-1}^{(1)}$

The purely elastic S-matrix associated to this algebra involves a set of $n-1$ particles with masses [10]

$$m_a = m\sin\left(\frac{\pi a}{n}\right),\ \ a = 1,2,\ldots,n-1. \tag{4.2}$$

The fusion angles are

$$u_{ab}^{a+b} = \frac{(a+b)\pi}{n} \quad (a+b<n), \qquad u_{ab}^{a+b-n} = \frac{(2n-a-b)\pi}{n} \quad (a+b>n), \tag{4.3}$$

and the minimal S-matrix elements are given by

$$\tilde{S}^{[ab]}(\theta) = \prod_{\substack{j=|a-b|+1\\ \text{step } 2}}^{|a+b|-1}\{j\}. \tag{4.4}$$

The affine Toda S-matrix is given by replacing $\{j\}$ with $\{j\}_0$ where the coupling constant $0<B<2$.

From both the minimal and affine Toda S-matrix, a supersymmetric S-matrix with $n-1$ particle super-multiplets can be built by appending the supersymmetric factors $S_{\rm P}^{[ab]}(\theta)$. In order to prove that this is consistent we must show that the supersymmetric part of the S-matrix satisfies the bootstrap equations which result from the fusing angles in (4.3). Before we can use the results in section 3.1 we must first take account of the fact that in this theory the particles are not self-conjugate, as assumed there, rather $\bar{a}=n-a$. Hence the crossing symmetry relation in (3.11) should be replaced by

$$G^{[ab]}(\theta)=G^{[n-a,b]}(i\pi-\theta). \tag{4.5}$$

It is easy to verify that the general solution (3.13) does satisfy the above relation with $H=n$.

Consider the bootstrap equations that arise from the fusing rules (4.3). If $a+b<n$, we have already shown that $S_{\rm P}(\theta)$ satisfies the corresponding bootstrap equation, namely (i) of (3.19). When $a+b\geq n$, the corresponding bootstrap equation is (schematically)

$$S_{\rm P}^{[d,a+b-n]}(\theta)\sim S_{\rm P}^{[da]}\left(\theta-\frac{i\pi(n-b)}{n}\right)S_{\rm P}^{[db]}\left(\theta+\frac{i\pi(n-a)}{n}\right). \tag{4.6}$$

Now using the fact that

$$S_{\rm P}^{[ab]}(\theta)\equiv S_{\rm P}^{[n-a,b]}(\theta), \tag{4.7}$$

which follows from (4.5) and (3.10), we can express the above as

$$S_{\rm P}^{[d,2n-a-b]}(\theta)\sim S_{\rm P}^{[d,n-a]}\left(\theta-\frac{i\pi(n-b)}{n}\right)S_{\rm P}^{[d,n-b]}\left(\theta+\frac{i\pi(n-a)}{n}\right). \tag{4.8}$$

But this is precisely the bootstrap equation (i) of (3.19) for $(n-a)+(n-b)\to(2n-a-b)$ which has already been verified. Hence the supersymmetric part of the S-matrix respects all the fusing of the bosonic theory and so the combined S-matrix $\tilde{S}^{[ab]}(\theta)S_{\rm P}^{[ab]}(\theta)$ is a consistent supersymmetric S-matrix.

4.2 *The case* $d_n^{(1)}$

The bosonic S-matrix related to this algebra describes a theory with n particles with masses [8]

$$\begin{aligned} m_s &= m_{s'}=m,\\ m_a &= 2m\sin\left(\frac{a\pi}{2(n-1)}\right)\qquad a=1,2,\ldots,n-2. \end{aligned} \tag{4.9}$$

Here we have labelled the particles $n-1$ and n as s and s', because they are associated to the spinorial spots of the d_n Dynkin diagram. We will refer to these two particles as the spinor and anti-spinor. For n even all the particles are self-conjugate, whereas for n odd $\bar{s}=s'$, and vice-versa.

The fusion angles between non-spinor particles are

$$
\begin{aligned}
&u_{ab}^{a+b} = \frac{(a+b)\pi}{2(n-1)} \quad (a+b \leq n-2), \qquad u_{ab}^{|b-a|} = \frac{(2(n-1)-|b-a|)\pi}{2(n-1)}, \\
&u_{ab}^{2(n-1)-a-b} = \frac{(a+b)\pi}{2(n-1)} \quad (a+b \geq n).
\end{aligned} \tag{4.10}
$$

Those involving the spinor particles are

$$
\begin{aligned}
u_{ss}^{a} = u_{s's'}^{a} = \frac{(n-1-a)\pi}{n-1}, \quad u_{sa}^{s} = u_{s'a}^{s'} = \frac{(n-1+a)\pi}{2(n-1)}, \\
u_{ss'}^{a} = \frac{(n-1-a)\pi}{n-1}, \quad u_{s'a}^{s} = u_{sa}^{s'} = \frac{(n-1+a)\pi}{2(n-1)}.
\end{aligned} \tag{4.11}
$$

In the first line of the above equation a is restricted to be even if n is even and a is odd if n is odd, and vice-versa in the second line.

The minimal S-matrix elements involving the non-spinor particles are

$$
\tilde{S}^{[ab]}(\theta) = \prod_{\substack{j=|a-b|+1 \\ \text{step } 2}}^{|a+b|-1} \{j\}\{2n-2-j\}. \tag{4.12}
$$

The elements for the scattering of a spinor particle with a non-spinor particle are

$$
\tilde{S}^{[sa]}(\theta) = \tilde{S}^{[s'a]}(\theta) = \prod_{\substack{j=0 \\ \text{step } 2}}^{2a-2} \{n-a+j\}. \tag{4.13}
$$

Finally, the S-matrix elements involving only spinor particles are

$$
\begin{aligned}
\tilde{S}^{[ss]}(\theta) = \tilde{S}^{[s's']}(\theta) = \prod_{\substack{j=1 \\ \text{step } 4}}^{2n-3} \{j\}, \quad \tilde{S}^{[ss']}(\theta) = \prod_{\substack{j=3 \\ \text{step } 4}}^{2n-5} \{j\}, \quad n \text{ even} \\
\tilde{S}^{[ss]}(\theta) = \tilde{S}^{[s's']}(\theta) = \prod_{\substack{j=1 \\ \text{step } 4}}^{2n-5} \{j\}, \quad \tilde{S}^{[ss']}(\theta) = \prod_{\substack{j=3 \\ \text{step } 4}}^{2n-3} \{j\}, \quad n \text{ odd.}
\end{aligned} \tag{4.14}
$$

As in the $a_{n-1}^{(1)}$ example the S-matrix of the affine Toda field theory is obtained by replacing $\{j\}$ with $\{j\}_0$ where the coupling constant $0 < B < 2$.

It is immediately apparent from the masses (4.9) that in this case the coupling constant H is fixed to be $2(n-1)$, the Coxeter number of d_n. In addition, we see that in the supersymmetric construction the spinor particles must be kink states, whereas the other states will be particles. So we introduce two sets of kinks: $\{K^{s}_{0,\frac{1}{2}}, K^{s}_{1,\frac{1}{2}}, K^{s}_{\frac{1}{2},0}, K^{s}_{\frac{1}{2}1}\}$ and $\{K^{s'}_{0,\frac{1}{2}}, K^{s'}_{1,\frac{1}{2}}, K^{s'}_{\frac{1}{2},0}, K^{s'}_{\frac{1}{2}1}\}$. The scattering of the kink degrees-of-freedom is governed by the kink S-matrix described in section 2.2.

We must now verify that the supersymmetric part of the S-matrix respects the bootstrap equations which follow from the fusing angles in (4.10) and (4.11). The first two cases in (4.10) follow immediately from (3.19). The bootstrap equation corresponding to the third fusing rule in (4.10) is

$$S_{\rm P}^{[d,2(n-1)-a-b]}(\theta) \sim S_{\rm P}^{[da]}\left(\theta - \frac{ib\pi}{2(n-1)}\right) S_{\rm P}^{[db]}\left(\theta + \frac{ia\pi}{2(n-1)}\right). \tag{4.15}$$

Although the non-spinor particle label runs from 1 to $n-2$ we can artificially extend the label to run to $2(n-1)$ by identifying a, with $1 \leq a \leq n-2$ with $2(n-1)-a$. This identification is consistent with S-matrix elements defined on the extended labels since

$$S_{\rm P}^{[ab]}(\theta) = S_{\rm P}^{[a,2(n-1)-b]}(\theta) = S_{\rm P}^{[2(n-1)-a,b]}(\theta) = S_{\rm P}^{[2(n-1)-a,2(n-1)-b]}(\theta). \tag{4.16}$$

The bootstrap equations corresponding to the first fusing rule in (4.10) are also satisfied by the S-matrices defined on the extended set of labels:

$$S_{\rm P}^{[d,a+b]}(\theta) \sim S_{\rm P}^{[da]}\left(\theta - \frac{ib\pi}{2(n-1)}\right) S_{\rm P}^{[db]}\left(\theta + \frac{ia\pi}{2(n-1)}\right). \tag{4.17}$$

Now if $a+b \geq n$ then by using (4.16) we see that (4.15) is satisfied.

Now we move on to consider scattering amplitudes containing kinks. Although we have introduced two sets of kinks associated to the spinor particles the situation is quite simple because the supersymmetric part of the S-matrix is blind to the labels s and s'. Bound-states corresponding to particles exist in the ss, ss' or $s's'$ channels with fusion angles given in (4.11). The associated bootstrap equations are satisfied by virtue of the discussion in sections 3.2 and 3.3. The important point to remember is that the supersymmetric part of the S-matrix satsifies the required bootstrap equation for it any value of the kink-kink fusing angle denoted ξ in section 3.2.

4.3 The pair $c_n^{(1)}$, $d_{n+1}^{(2)}$

This is the first example involving non-simply-laced algebras [8]. In these cases each of the S-matrices is associated to a pair of algebras. In addition, the S-matrices and the mass spectrum depend on a coupling constant, and there is no analogue of the minimal coupling constant independent S-matrix that exists in the simply-laced cases.

In the present case the mass spectrum is

$$m_a = 2m \sin\left(\frac{a\pi}{H}\right), \quad p = 1, 2, \ldots, n, \tag{4.18}$$

where the coupling constant $2n \leq H \leq 2n+2$. The particles are all self-conjugate and the fusion angles are

$$u_{ab}^{a+b} = \frac{(a+b)\pi}{H} \quad (a+b \leq n), \qquad u_{ab}^{|b-a|} = \frac{(H-|b-a|)\pi}{H}. \tag{4.19}$$

The S-matrix elements are

$$\tilde{S}^{[ab]}(\theta) = \prod_{\substack{j=|a-b|+1 \\ \text{step } 2}}^{a+b-1} [j]_0. \tag{4.20}$$

In this case the mass spectrum is exactly that of (3.8) and so we can build a supersymmetric S-matrix by simply appending the particle supersymmetric S-matrix of section 2.1 to the bosonic S-matrix, giving $\tilde{S}^{[ab]}(\theta) S_{\rm P}^{[ab]}(\theta)$.

4.4 The pair $b_n^{(1)}$, $a_{2n-1}^{(2)}$

In this case the spectrum contains n particles of mass [8]

$$m_a = 2m \sin\left(\frac{a\pi}{H}\right), \qquad a = 1, 2, \ldots, n-1, \qquad m_n = m. \tag{4.21}$$

As in the previous example, the quantity H can float in the region $2n-1 \leq H \leq 2n$. The particles are all self-conjugate and the fusion angles are

$$\begin{aligned} u_{ab}^{a+b} &= \frac{(a+b)\pi}{H} \quad (a+b<n), \qquad u_{ab}^{|b-a|} = \frac{(H-|b-a|)\pi}{H} \\ u_{nn}^{a} &= \frac{(H-2a)\pi}{H},\ u_{na}^{n} = \frac{(H/2+a)\pi}{H}. \end{aligned} \tag{4.22}$$

The S-matrix elements are

$$\begin{aligned} \tilde{S}^{[ab]}(\theta) &= \prod_{\substack{j=|a-b|+1 \\ \text{step } 2}}^{a+b-1} [j]_0 \\ \tilde{S}^{[an]}(\theta) &= \prod_{\substack{j=1 \\ \text{step } 2}}^{2a-1} \{H/2-a+j\}_0 \\ \tilde{S}^{[nn]}(\theta) &= \prod_{\substack{j=1-n \\ \text{step } 2}}^{n-1} \{H/2-j\}_{-1/4}. \end{aligned} \tag{4.23}$$

From the mass formula, it is clear that the particle n, corresponding to the spinor spot of the b_n Dynkin diagram, should be associated to a kink multiplet in the supersymmetric theory. The remaining particles become ordinary particle supermultiplets in the supersymmetric theory. The bootstrap equations involving just these particles follows from the results of section 3.1. The bootstrap equations involving the kinks are verified in the same way as in the $d_n^{(1)}$ theory.

5. Other theories

In this section we briefly consider the S-matrices of other well-known supersymmetric integrable theories.

5.1 The O($2n$) supersymmetric sigma model

The lagrangian density of the supersymmetric O($2n$) model is [11]

$$\mathcal{L} = \frac{1}{2g}\left[(\partial_\mu n_a)^2 + i\bar{\psi}_a \partial\!\!\!/ \psi_a + \tfrac{1}{4}\left(\bar{\psi}_a\psi_a\right)^2\right], \tag{5.1}$$

where n_a and ψ_a are a $2n$-component real scalar field and Majorana fermion, respectively, satisfying the constraints $n \cdot n = 1$ and $n \cdot \psi = 0$. The theory (5.1) has a global O($2n$) symmetry and a global $N = 1$ supersymmetry. Notice that the bosonic part of the theory is just the O($2n$) sigma model, the fermionic part is the O($2n$) Gross-Neveu model, and the coupling between the bosons and fermions is due solely to the constraint.

The S-matrix conjectured by Shankar and Witten [11] for the scattering of the fundamental states, which transform in the vector representation of O(N) and form a particle super-multiplet of supersymmetry, has the factored form of (1.1). In this case the bosonic factor $\tilde{S}(\theta)$ is the S-matrix of vector state of the O($2n$) Gross-Neveu model [12,13] and the supersymmetric part is precisely the S-matrix block of a particle super-multiplet written down explicitly in section 3.1.

The outstanding question is what is the full S-matrix of the theory generated by the bootstrap programme. Since the supersymmetric part introduces no additional poles onto the physical strip it is natural to suppose that the full S-matrix enjoys the same fusing angles as the O($2n$) Gross-Neveu model. This theory has a spectrum and fusing date which is exactly that of the $d_n^{(1)}$ model discussed in section 4.2, where the particle of mass m_1 transforms in the vector representation and the heavier particles m_a, for $a = 2, \ldots, n-2$, transform in reducible representations. The two 'spinor' particles transform in the spinor and anti-spinor representations of O($2n$). In the supersymmetric theory the particles with masses m_a, for $a = 1, \ldots, n-2$, become particle super-multiplets, while the spinor particles become kink super-multiplets.

The fact that the resulting S-matrix satisfies the bootstrap equations, follows from the discussion in section 4.2. The S-matrix conjectured by Shankar and Witten for the scattering of the vector states has been subjected to a highly non-trivial test using the Thermodynamic Bethe Ansatz (TBA) [14]. Not only is the conjectured S-matrix completely consistent with the TBA calculation but as a by-product an exact value for the mass-gap of the model is found (see [15] and references therein). The case $n = 2$ deserves a special mention because in this case the theory is actually the supersymmetric SU(2) principal chiral model. It has been argued that the spectrum of this theory should include only the spinor states, i.e. only the two kink super-multiplets which transform as $(2, 1) + (1, 2)$ of the SU(2)×SU(2) symmetry. Our S-matrix is valid in this case since the poles corresponding to the particle super-multiplets disappear from the physical strip and the S-matrix of the kink states is consistent by itself. This model is the first of the SU(n)

series of supersymmetric principal chiral models whose S-matrices have been written down in [16] and subjected to the highly non-trivial TBA test.

5.2 *The supersymmetric sine-Gordon theory*

The exact S-matrix of the supersymmetric sine-Gordon theory [4,11] has been conjectured in [3]. The spectrum of the model consists of a soliton and anti-soliton, of mass m, which both transform as kink super-multiplets. Just as in the sine-Gordon theory without supersymmetry, the soliton and anti-soliton have bound-states called breathers of mass

$$m_a = 2m \sin\left(\frac{a\gamma}{16}\right), \tag{5.2}$$

where $a = 1, 2, \ldots, N < 8\pi/\gamma$, and γ is a funcion of the coupling constant of the model. The breathers transform in particle super-multiplets.

The S-matrix has the factored form of (1.1) where the bosonic factor is the S-matrix of the sine-Gordon theory [12]. In order to prove that the full S-matrix satisfies the bootstrap equations we must consider the possible fusing rules. Denoting the soliton by s and the anti-soliton by s' the fusing rules are

$$u_{ab}^{a+b} = \frac{(a+b)\gamma}{16} \quad (a+b \leq N), \qquad u_{ab}^{|b-a|} = \pi - \frac{|b-a|\gamma}{16}, \tag{5.3}$$

for processes involving the breathers alone and

$$u_{ss'}^{a} = \pi - \frac{a\gamma}{8}, \qquad u_{sa}^{s} = u_{s'a}^{s'} = \frac{\pi}{2} - \frac{a\gamma}{16}. \tag{5.4}$$

for processes involving the solitons. The fusing angles of the breathers are identical to those in (3.9) and the bootstrap equations follow from the results derived in section 3.1 summarized in (3.19). For the solitons the discussion in section 3.2 implies that the bootstrap equations are satisfied.

6. Discussion

The fact that the S-matrices of many integrable field theories are known exactly follows from the tractability of the bootstrap equations. Remarkably, the bootstrap data encoded in the fusing angles are related to (affine) Lie algebras. It is surprising that when one considers integrable theories with $N = 1$ supersymmetry that solutions of the bootstrap can be associated to exactly the same fusing data as for the bosonic theories. Using this fact we have been able to show that some well-known supersymmetric integrable theories, namely the supersymmetric O($2n$) sigma model and the supersymmetric sine-Gordon model, have exact S-matrices which complete the bootstrap programme.

In addition we showed how supersymmetric S-matrices could be associated to the theories with the same fusing data as the affine Toda field theories. We do not know whether these S-matrices describe any Lagrangian field theories. For instance they cannot describe the usual supersymmetric Toda theories, since these are associated to super Lie algebras, rather than Lie algebras [17].

It is important for us to stress that the solutions of the bootstrap equations for theories with $N = 1$ supersymmetry that we have found are not exhaustive. For instance, recently, the S-matrices of the $N = 1$ supersymmetric SU(n) principal chiral models have been found [16]. These S-matrices generalize the kink S-matrices described here since they are constructed from solutions of the Yang-Baxter equation for an SU(n) quantum group. As we have already mentioned, for $n = 2$ this model is the supersymmetric O(4) model.

TJH is supported by a PPARC Advanced Fellowship.

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